Shapes of Quadratics

Concept Lesson Guide

LESSON OVERVIEW

Overview: This is a set of related lessons, with structured exploration of graphs of different forms of quadratic functions in Investigations 1 and 2, application of the discoveries from these investigations to write equations for specific graphs in Investigation 3, and discussion of conclusions in the Wrap-Up. In Investigations 1 and 2, students first review the graph of the basic quadratic function $y = x^2$, then systematically explore the graphs of 6 forms of quadratic equations and their relationship to the graph of the basic function. Students use graphing calculators to systematically examine how the form of the function and changes in coefficients and constants affect the appearance of the graph and the location of its vertex on the coordinate plane. In Investigation 1, students explore equations of the form $y = ax^2$, $y = x^2 + c$ and $y = ax^2 + c$, then discuss the effects of the values of a and c on the appearance and location of the graph. In Investigation 2, students explore equations of the form $y = (x + h)^2$, $y = a(x + h)^2$, and $y = a(x + h)^2 + k$. This is followed by a class discussion in which selected groups present their results and summarize the effects of form and values of coefficients and constants on the graph of quadratic functions.

In Investigation 3, students "reverse" their thinking in Investigations 1 and 2: In those investigations students went from equations to graphs; in Investigation 3, students go from graphs to equations. Students play a game the object of which is to write an equation that describes a "mystery graph". Students predict an equation that corresponds to a given graph, then revise that equation until they determine the graph that matches the equation. After playing the game, they reflect on how they revised their equation to correct specific differences between the mystery graph and the graph their equation produced.

In the Wrap-Up, students reflect on what they learned in the Investigations, and discuss how the form of the quadratic equation and the values of the coefficients and constants affect the appearance of the graph and its location in the coordinate plane.

One of the risks in exploring the relationship between equations and their graphs is that students will discover patterns in a few special cases that do not hold in general. For example, in $y = ax^2 + c$, students will discover that c is the y-intercept **and** that it denotes the magnitude of the vertical translation of the graph. However, **only** the discovery that c is the y-intercept is true for the general quadratic equation, $y = ax^2 + bx + c$. When $b \neq 0$, c does *not* represent the vertical translation of the graph. (Question 7 in the Wrap-Up explicitly addresses this possible misconception.) Even for correct generalizations, patterns may be observed without attention to the underlying explanations of why these patterns occur. Thus, it is especially important to follow explorations with careful class discussions in which observations are shared **and** the underlying reasons for them are made explicit.

Investigation 1: From Equations to Graphs I $y = ax^2$ $y = x^2 + c$ $y = ax^2 + c$	**Investigation 2:** From Equations to Graphs II $y = (x + h)^2$ $y = a(x + h)^2$ $y = a(x + h)^2 + k$	**Investigation 3:** From Graphs to Equations ***Name that Graph***	**Wrap-Up** Pulling it All Together

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NCTM Standards Addressed:
- Explore relationships between symbolic expressions and graphs of lines …
- Build new mathematical knowledge through problem solving.
- Make and investigate mathematical conjectures.

Mathematical Goals of the Lesson:
- Recognize the characteristic shape of a quadratic function and describe its maximum or minimum points (vertex).
- Develop an understanding of the relationship between quadratic equations and their graphs.
- Develop an understanding of the effect of coefficients and constants in quadratic equations on the appearance and position of their graphs.
- Predict from quadratic equations whether quadratic functions have maximum or minimum values and their location on the coordinate plane.
- Write quadratic equations to describe specific graphs.
- Reason mathematically and use and make connections among a variety of mathematical representations.

Academic Language Goals of the Lesson:
- Develop academic vocabulary to be used in the descriptions.
- Describe algebraic patterns orally or in writing.
- Explain the process used in solving the task, orally or in writing.
- Use appropriate mathematical language in all explanations and discussions.

Assumption of Prior Knowledge:
- Experience graphing the basic quadratic equation, $y = x^2$.
- Experience using graphing calculators to explore how different values of m and b affect the graph of linear equations in the form $y = mx + b$.

Academic Language:
- Characteristics of a graph
- Minimum and maximum values
- Vertex
- Parabola
- Wide/narrow graph
- Wide/narrow parabola
- Axis of symmetry (optional)
- Intercepts
- Quadratic function
- Quadratic equation
- Constant
- Coefficient
- Absolute value
- Translation (optional)
- Open up (concave up), open down (concave down)
- Stretch, compress
- Range of values

Materials:
- Lesson Task
- Graphing calculators (1 per student) plus view screen (1 for teacher)
- Name That Graph Recording Sheet (one per student)
- Investigation 1 and 2 Recording Sheets (optional)
- Graph paper recording sheets (optional Investigation 1 and 2 Recording Sheets)

INVESTIGATION 1

FROM EQUATIONS TO GRAPHS I

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Key:
Suggested teacher questions are shown in bold print.
Questions and strategies that support ELLs are underlined and identified by an asterisk."
The rationale for selected ELL strategies are in boxes in the margin.
Possible student responses are shown in italics.

Phase	SET-UP PHASE: Setting Up the Mathematical Task — Investigation 1
S E T U P S E T U P S E T U P	**INTRODUCING THE TASK** • Start by introducing the goal of the lesson: "In this lesson, you will investigate how you can use the graph of the basic quadratic function $y = x^2$ to describe the graphs of more complex quadratic functions." • Begin with "Getting Started". Ask students to state the basic quadratic equation and enter it into their graphing calculators. • Ask them to describe the characteristics of the graph of $y = x^2$ and discuss them with members of their group. Listen for terms describing both appearance and location of the vertex, e.g., symmetric, opens up, minimum value at the origin (or (0,0)), same on both sides of the y-axis, no maximum value. • Debrief the characteristics of the basic graph with the entire class. Ask each group to give one characteristic. Take advantage of opportunities to interject mathematical vocabulary (e.g., line of symmetry, symmetric; vertex, y-intercept, x-intercept, etc.). Be sure to remind them that the graph is a parabola. Also make explicit that the vertex is the origin, and that it is both the x- and y-intercept. • First have students read Investigation 1 silently (individually), then select a student to read it aloud as others follow along. Ask students to circle words that they do not understand, and make note of these words as you circulate around the room.* • Ask a student to read Investigation 1 out loud as others follow along. • Ask several students to explain what they think they are being asked to do. Be sure to ask, "What is a coefficient? What is a constant?" • Specifically ask, "Exactly how are you going to use your graphing calculator? What range of values will you use? What window will you use? What will you record?" • Try an example with the whole class for $y = ax^2$ ***for one value of a.*** Be sure students enter the basic equation as the first equation in the "Y=" list. Be sure to reinforce the term "coefficient" by using it in this example, "What value should we use for the coefficient *a*?" Clarify any confusions students may have but do not suggest specific values for their investigation.

It is important to give ELLs the opportunity to read the text silently before following along. By circling words they do not understand, the teacher can know the additional vocabulary words that need to be developed in order to provide access to the mathematics in the task.

To assist ELLs' participation in the class discussion:*

- Allow time for students to first talk in small groups (pairs) and then have the groups report to the whole class.*
- Reinforce appropriate language as students communicate their ideas (e.g., revoice a student's contribution in complete, grammatically correct language). Ask students if you have captured what they said.*
- Create work groups that are heterogeneous according to language proficiency.*
- Model appropriate mathematical language, emphasizing vocabulary used in appropriate context.*

Phase	EXPLORE PHASE: Supporting Students' Exploration of the Task STRUCTURE
E X P L O R E	***Use this structure for both Investigation 1 and Investigation 2.*** **<u>PRIVATE THINK TIME</u>** • Ask students to work individually for about 15 minutes so that they can make sense of the problem for themselves. The students in each group should determine how they want to "share the workload", (e.g., some groups may decided that each person should try one or two numbers greater than 1, one or two between 0 and 1, one or two between -1 and 0, and one or two less than -1, etc. Another group may decide that one person should investigate numbers greater than 1, another, numbers between 0 and 1, a third, numbers between 0 and -1, and the fourth, numbers less than -1. By investigating possible values in a systematic way, students will identify patterns more easily. • Each student should explore the three equation forms individually and <u>write*</u> conjectures about how the values of the coefficients and constants affect the appearance and location of the vertex of the graphs. • Circulate around the classroom and clarify confusions. Be careful to NOT give away too much information or suggest conjectures.

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	<u>SMALL-GROUP WORK</u>
	• After about 15 minutes, ask students to work with their partner or in their small groups to discuss what they discovered about each form of the quadratic equation.
	• As students are working, circulate around the room.
	  o Be sure students use a sufficient range of values, including ones between 0 and 1, 0 and -1, and less than or equal to -1.
	  o Be persistent in asking questions related to the mathematical ideas, exploration strategies, and connections between representations.
	  o Be persistent in asking students to explain their thinking and reasoning.
	  o Be persistent in asking students to explain, in their own words, what other students have said.
	  o <u>Be sure to model appropriate mathematical language*</u> and be persistent in asking students to use appropriate mathematical language.
	
	<u>What do I do if students have difficulty getting started?</u>
	• Ask: "What is a possible value for a (c, h, k)?" "What equation do you get when you use that value?" "How would you describe your graph? Compare your graph to that of $y = x^2$. How are they alike? How are they different?" "Now, try another value for (a, c, h, k)."
	
	<u>What do I do if students finish early?</u>
	• Look at students' work (list of values, equations) and be sure that they have used a sufficient range of values for coefficients and constants.
	• Groups that complete Investigation 1 can go on to Investigation 2. If some groups complete both investigations before it is time for the SDA phase, they can read through the instructions for the game and try it out on their own, making and testing some of their conjectures.

Strategies for language development (in this case writing skills) need to be made more specific. <u>Writing</u> the conjectures will establish a record of what the students are thinking.

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Phase	EXPLORE PHASE: Supporting Students' Exploration of the Task STRUCTURE (continued)

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	<u>MONITORING STUDENTS' RESPONSES</u>
	• As you circulate, attend to students' mathematical thinking and to their conjectures, in order to identify those responses that will be shared during the Investigation 1 and Investigation 2, Share, Discuss, and Analyze Phases. (Investigation 1 will be discussed before the entire class begins Investigation 2, though groups that finish Investigation 1 may start Investigation 2 prior to the whole-class discussion.)
	• During this phase, groups will (1) present the equations they tried, (2) their corresponding graphs, and (3) conclusions about how the values of coefficients and constants affect the appearance and location of the vertex of the graph.
	• In the interest of time, have only one group present its results for each graph form. Give other groups the opportunity to comment, ask questions, etc.
	• For this task, you will need to:
	  o Identify groups to present their result for $y = ax^2$, $y = x^2 + c$ and $y = ax^2 + c$, respectively, Investigation 1; and $y = (x + h)^2$, $y = a(x + h)^2$ and $y = a(x + h)^2 + k$, respectively, (Investigation 2).
	  o Look for groups that tried an interesting range of values. For example, for Investigation 1, have wide and narrow graphs that open up and down, and have a variety of vertical translations. For Investigation 2, the vertex should appear in all four quadrants, along with variation in vertical translations and having minimum and maximum values.
	  o If possible, each group should present their results using a graphing calculator. Students can either display a subset of the equations they used and the resulting graphs by connecting a calculator of one group member to the view screen. Or, if that is not possible, they can enter the equations they used into the calculator that the teacher has connected to the view screen.
	
	Alternatively, students could record each of their equations and graphs on the graph paper recording sheet. (Use one sheet for each equation form.) Work could be presented with a paper projection device, or each group could make a transparency of the equation form they will present.

Phase	INVESTIGATION 1 QUESTIONS RE: $y = ax^2$, $y = x^2 + c$ and $y = ax^2 + c$ 1. How are the graphs similar to, and different from, the graph of the basic function $y = x^2$? 2. How are these graphs, similar to, and different from, each other? 3. How do changes in the values of *a* and *c* affect: the appearance of the graph? the location of the vertex of the graph in the coordinate plane?*			
	Possible Solutions	**Possible Questions**	**Misconceptions/ Errors**	**Questions to Address Misconceptions/Errors**
E X P L O R E	Look for indictors of students' effective exploration: • Trying a range of values for *a* and *c*. • Graphing each equation and comparing the graph to that of the basic function. • Making conjectures about how the values of *a* and *c* affect the appearance and location of the vertex of the graph.	**Ask questions such as:** • **What happens to the graph when *a* is negative?** • **What happens to the graph when *a* is between 0 and 1?** • **How can you tell if a parabola will open downward or open upward?**	• Values of *a* between 0 and 1 and 0 and -1 make a parabola narrower (i.e., smaller); larger values for *a* make the parabola wider (i.e., bigger).	• **Show me a parabola that has a _____ value (*e.g., large value, value between 0 and 1*) for *a* (or *c*). What does it look like?**
E X P L O R E E X P L O R E	Look for indicators of students' understanding: • Concluding that if *a* is negative, the graph opens down; if *a* is positive, the graph opens up. (Concave down; concave up) • Concluding that if $\lvert a\rvert > 1$, the graph is narrower; if $0 < \lvert a\rvert < 1$, the graph is wider. • Concluding that *c* produces a vertical translation of c units; a translation up when *c* is positive, down when c is negative.	• **How can you make a parabola that is wider than the one you just made? How can you make a parabola that is more narrow? Try it out and see if your conjecture works.** • **How would you write the equation of a parabola whose vertex has a negative y-value?** • **What do parabolas with maximum (minimum) values look like? of the parabolas you graphed?**	• When *a* is negative, the parabola has a minimum value (because negative numbers are smaller).	• **In what direction does the parabola open when *a* is negative? Does that give a maximum or minimum value?**

Many students (in addition to ELLs) have difficulty organizing their thoughts when it comes to "similarities or differences". Have students set up a "similarities and differences" table so that they can organize their observations.*

To help students understand complex questions it is often helpful to break them into their component parts. Students may also want to use an organizational chart to record their conclusions (see a sample chart on page 13).*

Phase	INVESTIGATION 1 QUESTIONS RE: (continued) $y = ax^2$, $y = x^2 + c$ and $y = ax^2 + c$ 1. How are the graphs similar to, and different from, the graph of the basic function $y = x^2$? 2. How are these graphs, similar to, and different from, each other? 3. How do changes in the values of *a* and *c* affect: <u>the appearance of the graph? the location of the vertex of the graph in the coordinate plane</u>?*			
	Possible Solutions	**Possible Questions**	**Misconceptions/Errors**	**Questions to Address Misconceptions/Errors**
E X P L O R E E X P L O R E	• Combining what they learned from investigating $y = ax^2$, $y = x^2 + c$ to predict what will happen with $y = ax^2 + c$ even before trying equations of this form. • Correct use of mathematics vocabulary, (e.g., maximum, minimum, vertex, y-intercept).	**Ask questions such as:** • **Predict what the graph of a form $y = ax^2 + c$ parabola would look like. What do you think your conclusions about $y = ax^2$, $y = x^2 + c$ can tell you about $y = ax^2 + c$? Test your conjecture and see if it works.** • **What are the y-intercepts of the parabolas you graphed?** • **What is the relationship between the y-intercept and vertex? Will that always be true? How do you know?** • <u>**Explain in your own words what _______ (another student) said.***</u>	• When parabolas are translated up or down, they also become wider or narrower[1] (i.e., students think that a parabola that opens up becomes skinnier when it is translated up because they see only the part of the graph that is closer to the vertex).	• **Look at the (x,y) values for $y=x^2$ and $y=x^2 + c$. How are they alike? How are they different? (Might suggest putting values in a table that shows x, x^2, $x^2 + c$.)** • **When the parabola was translated up, are we able to see as much of the graph in our window? What do we have to do to our window to see as much of the graph as we saw before it was translated?**

This should become a common practice in your classroom.

The graphs can be misleading because students tend to focus on the distance from one curve to another, which appears to change with different x values, rather than the vertical differences associated with different y values for the same x value." Contemporary Mathematics in Context, TE Volume 2, p. T275.

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Share, Discuss, and Analyze - Investigation 1

Orchestrating the mathematical discussion: a possible Sequence for sharing student work, Rationale and Mathematical Ideas to achieve the goals of the lesson, and possible Student Responses that demonstrate understanding.

The purpose of this sharing/discussion is to make explicit the conclusions from Investigation 1.

Phase	Sequencing of Student Work	Rationale and Mathematical Ideas	Possible Questions and Student Responses
SHARE DISCUSS ANALYZE	Order of group presentations: 1. Conclusions from exploring equations of the form $y = ax^2$	This discussion will make explicit how values of *a* and *c* affect the graph of quadratic equations of forms $y = ax^2$, $y = x^2 + c$ and $y = ax^2 + c$ • If *a* is negative, the graph opens down; if *a* is positive, the graph opens up. (Concave down; concave up) When *a* is negative, the function has a maximum value; when *a* is positive, it has a minimum value. • If $\|a\| > 1$, the graph is narrower; if $0 < \|a\| < 1$, the graph is wider. Visually, this is a stretch of y values if $a > 1$; a compression if $0 < a < 1$. (Stretch or compression may help address the misconception that $a > 1$ makes the graph wider; $0 < a < 1$ makes the graph narrower.)	What values did you use for *a*? (*c*?) • *We tried lots of different values: large numbers, negative numbers, and fractions.* What effect did different values of *a* have on the graphs? (If necessary, prompt by different values, e.g., What happened when *a* was positive? What direction did it open? What happened when *a* was negative?) • *If a is negative, the graph opens down. If a is positive, the graph opens up.* Why does a negative *a* value cause the parabola to be reflected? Why does an *a* value between -1 and 1 cause the graph to become wider, while a value greater than 1 or less than -1 cause the graph to become more narrow? (After each group presentation, give other groups an opportunity to comment, ask questions, <u>restate the conclusions given</u>* or expand upon the conclusions presented. For example:
	2. Conclusions from exploring equations of the form $y = x^2 + c$	• *c* produces a vertical translation of c units; a translation up when *c* is positive, down when *c* is negative. • *c* is the y-intercept of the parabola. • The vertex of the parabola is (0, c).	▪ ___, in your own words tell us what ___ concluded about the effect of 'a' on the graphs of parabolas. ▪ Do you agree with Group X's conclusions? Why or why not? Did you discover something different than what ____ did? Explain. So we talked about positive, negative, and fractional values of *a*. Can *a* ever be 0? Why or why not? We talked about how the graph of the parabola changes when *a* changes. Does anything stay the same? Why does it stay the same?

Phase	Sequencing of Student Work	Rationale and Mathematical Ideas	Possible Questions and Student Responses
S H A R E D I S C U S S A N D A N A L Y Z E			Explain the effect that different values of *c* have on the graphs. If necessary, prompt by different values. For example: What happened when *c* was positive? What direction did it shift? What happened when *c* was negative? ▪ *If c is negative, the graph shifts down. If c is positive, the graph shifts up.* ▪ ___, in your own words tell us what ___ concluded about the effect of *c* on the graphs of parabolas. ▪ Do you agree with Group X's conclusions? Why or why not? Did you discover something different than what ____ did? Explain.
	3. Conclusions from exploring equations of the form $y = ax^2 + c$	Combining the effects of *a* and *c* in a single equation. Students should come to understand that *a* and *c* affect the graph independently. It is important for students to understand that the y-intercept of a graph is the value of the function at x = 0.	What do you think happens to the graph of a parabola when you change both *a* and *c*? Without using your calculator, what do you predict the graph of $y = ½ x^2 – 3$ would look like? Draw the graph in your mind. What does it look like? Why do you think it looks like that? Now, without using your calculator, what do you predict the graph of $y = -3x^2 + ½$ would look like? Draw the graph in your mind. What does it look like? Why do you think it looks like that? Check your answers using the calculator. What do you notice? When graphing $y = ax^2 + c$, does it matter if you consider the effect *a* has on the graph first and then consider the effect the *c* value has or could you consider *c* first and then *a*? Explain what you mean. What relationship is there between the y-intercept on the graph of the parabola and the equation? How could you verify this algebraically?

INVESTIGATION 2

FROM EQUATIONS TO GRAPHS II

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<table>
<tr><td>Phase</td><td colspan="4">INVESTIGATION 2 QUESTIONS RE: $y = (x + h)^2$, $y = a(x + h)^2$ and $y = a(x + h)^2 + k$
1. How are the graphs similar to, and different from, the graph of the basic function $y = x^2$?
2. How are these graphs similar to, and different from, each other?
3. How do changes in the values of a, h, and k affect the: <u>appearance of the graph?</u> <u>the location of the vertex of the graph in the coordinate plane?</u> *</td></tr>
<tr><td></td><td>Possible Solutions</td><td>Possible Questions</td><td>Misconceptions/Errors</td><td>Questions to Address Misconceptions/Errors</td></tr>
<tr><td>E X P L O R E

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E X P L O R E</td><td>Look for indictors of students' effective exploration:
• Trying a range of values for a, h and k.
• Systematically selecting values – (i.e., changing the value of one at a time).
• Graphing each equation and comparing the graph to that of the basic function.
Look for indicators of students' understanding:
• Concluding that h produces a horizontal translation of h units; a translation to the right when h is negative, to the left when h is positive.
• Applying what was discovered in Investigation 1 about the effect of a to graphs of equations with of the form $y = a(x + h)^2$ and $y = a(x + h)^2 + k$.
• Concluding that k produces a vertical translation of k units; a translation up when k is positive, down when k is negative.</td><td>Ask questions such as:
• Have you tried negative numbers?
• Have you tried numbers between 0 and 1?
• Do you have parabolas that shift to the right and that shift to the left?
• How can you make a wider (narrower) parabola?
• Does the vertex of any parabola have a negative y value?
• What do parabolas with maximum (minimum) values look like?</td><td>• Values of a between 0 and 1 and 0 and -1 make a parabola narrower (i.e., smaller); larger values for a make the parabola wider (i.e., bigger).
• When a is negative, the parabola has a minimum value (because negative numbers are smaller).
• When parabolas are translated up or down, they also become wider or narrower.[2]
• When h is positive, the parabola shifts to the right (positive direction); when negative, shifts to the left (negative direction).</td><td>• Show me a parabola that has a <u>(e.g., large value, value between 0 and 1)</u> value for a (or c). What does it look like?
• In what direction does the parabola open when a is negative? Does that give a maximum or minimum value?
• Look at the (x, y) values for $y=x^2$ and $y=x^2 + c$. How are they alike? How are they different? (Might suggest putting values in a table that shows x, x^2, $x^2 + c$.)
• In what direction does the parabola shift when h is negative? When h is positive?
• Look at the (x, y) values for $y=x^2$ and $y=(x + h)^2$. How are they alike? How are they different? (Might suggest putting values in a table that shows x, x^2, $(x + h)^2$.)</td></tr>
</table>

[2] "The graphs can be misleading because students tend to focus on the distance from one curve to another, which appears to change with different x values, rather than the vertical differences associated with different y values for the same x value." Contemporary Mathematics in Context, TE Volume 2, p. T275.

Phase	**INVESTIGATION 2 QUESTIONS RE:** $y = (x + h)^2$, $y = a(x + h)^2$ and $y = a(x + h)^2 + k$ 4. How are these graphs similar to, and different from, the graphs of the equations in Investigation 1? Specifically, how are the effects of the values of *a* and *c* in the equations in Investigation 1 similar to, or different from, the effects of *a*, *h*, and *k* in the equations above?			
EXPLORE	**Possible Solutions**	**Possible Questions**	**Misconceptions/ Errors**	**Questions to Address Misconceptions/ Errors**
EXPLORE	Look for indictors of students' effective exploration: • Making conjectures about how the values of *a*, *h*, and *k* affect the appearance and location of the vertex of the graph. Look for indicators of students' understanding: • Combining what they learned from investigating $y = (x + h)^2$, $y = a(x + h)^2$ and Investigation 1 to predict what will happen with $y = a(x + h)^2 + k$ even before trying equations of this form. • Correct use of mathematics vocabulary, (e.g., maximum, minimum, translation). • Noticing that the vertex is **not** the y-intercept for the equations in Investigation 2 (unless h = 0), in contrast to those in Investigation 1.	**Ask questions such as:** • **What do your conclusions about equations 2, 3, 4 and 5 tell you about equation 6?** • **What are the y-intercepts of the parabolas you graphed?** • **What is the relationship between the y-intercept and vertex?** • **<u>Explain in your own words what ___ said (another student)</u>.***		

Questions that ask students to analyze situations on the basis of similarities and differences require teachers to provide structure. Ask students to set up a table (provide the framework) for them to respond to these questions. For example (see below):

Possible Organizer:

Function	Appearance		Location of vertex	
	Similarities	Differences	Similarities	Differences

Share, Discuss, and Analyze - Investigation 2

Orchestrating the mathematical discussion: a possible Sequence for sharing student work, Rationale and Mathematical Ideas to achieve the goals of the lesson, and possible Student Responses that demonstrate understanding.

The purpose of this sharing/discussion is to make explicit the conclusions from Investigation 2 (i.e., equations of the form $y = (x + h)^2$, $y = a(x + h)^2$ and $y = a(x + h)^2 + k$).

Phase	Sequencing of Student Work	Rationale and Mathematical Ideas	Possible Questions and Student Responses
SHARE **DISCUSS** **ANALYZE**	Order of group presentations: 1. Conclusions from exploring $y = (x + h)^2$ 2. Conclusions from exploring $y = a(x + h)^2$	This discussion will make explicit how values of *a, h* and *k* affect the graph of quadratic equations of Investigation 2. • *h* produces a horizontal translation of *h* units; a translation to the right when *h* is *negative*, to the left when h is *positive.* • replacing *x* by *x* + *h* is like moving the plane (the coordinates) *h* units to the right "under the graph" of the function, so it is as if the graph had moved *h* units to the left. • If *a* is negative, the graph opens down; if *a* is positive, the graph opens up. (Concave down; concave up) When *a* is negative, the function has a maximum value; when *a* is positive, it has a minimum value. • If |*a*| > 1, the graph is narrower; if 0 < |*a*| < 1, the graph is wider. Visually this is a stretch of y values if *a* > 1; a compression if 0 < *a* < 1. (Stretch or compression may help address the misconception that |*a*|>1 makes the graph wider; 0 < |*a*| < 1 makes the graph narrower.) Students will see this more clearly in a table of values.	**What values did you use for *h*? (*a*, *k*?)** • *We tried lots of different values: large numbers, negative numbers, and fractions.* (After each group presentation, give other groups an opportunity to comment, ask questions, or expand upon the conclusions presented. For example: **<u>Each group should write down (on a card or piece of paper) a question that you would like to ask Group X.*</u>** **Do you agree with Group X's conclusions? Why or why not?** **Did you discover anything different?** **What effect did different values of *h* have on the graphs?** (If necessary, prompt by different values, e.g., what happened when *h* was positive? What direction did it shift? What happened when *h* was negative?) **How many units did the graph shift?** • *If h is negative, the graph shifts to the right. If h is positive, the graph shifts to the left.* • *The graph shifted h units to the right or left.* NOTE: The direction of the horizontal translation is somewhat counter-intuitive. Students often think that positive values of *h* shift the graph in the positive direction (to the right) and negative values, in the negative direction (to the left). One way to address this is to make a table that compares the values of x, x^2, $(x + h)^2$. (A horizontal table—with values for x, x^2, and $(x + h)^2$ as the rows for a particular value of *h*.) Even with a table, the reason underlying the relationship may still not be clear. It is not necessary to push for an explanation at this point. Making a conjecture based on observed patterns is sufficient for now. This relationship will be addressed again in Unit 4, when students are also solving quadratic equations and relating their zeros and x-intercepts.

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Phase	Sequencing of Student Work	Rationale and Mathematical Ideas	Possible Questions and Student Responses
SHARE DISCUSS ANALYZE	3. Conclusions from exploring $y = a(x + h)^2 + k$	• Applying what was discovered in Investigation 1 about the effect of *a* to graphs of equations of forms $y = a(x + h)^2$ and $y = a(x + h)^2 + k$. • Concluding that *k* produces a vertical translation of k units; a translation up when *k* is positive, down when *k* is negative. • Combining what they learned from investigating $y = (x + h)^2$, $y = a(x + h)^2$ and Investigation 1 to predict what will happen with $y = a(x + h)^2 + k$ even before trying equations of that form. • Combining the effects of *a* and *k* in a single equation.	**What effect did different values of *a* have on the graphs?** **Did you explore this? Or did you apply what you noticed in Investigation 1?** (see above) **What effect did different values of *k* have on the graphs?** (If necessary, prompt by different values. For example: What happened when *k* was positive? What direction did it shift? What happened when *k* was negative?) **How many units did the graph shift?** ▪ *If k is negative, the graph shifts down. If k is positive, the graph shifts up. It shifts up or down k units.* ▪ *k is just like c. It moves the graph up and down.* **How are the effects of the values of a and *c* in equations in Investigation 1 similar to, or different from, the effects of *a*, *h*, and *k* in the equations in Investigation 2?** (see above) What conclusions did you reach from your graphs? (see above)

INVESTIGATION 3

FROM GRAPHS TO EQUATIONS

NAME THAT GRAPH

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Investigation 3: Play the game *Name that Graph*.

Phase	SET-UP PHASE: Setting Up the Mathematical Task — Investigation 3
S E T U P S E T U P S E T U P	This game provides an opportunity for students to practice what they learned in Investigations 1 and 2 about the effect of coefficients and constants on the graphs of quadratic equations. In the investigations, students worked from equations to graphs; in this game, students reverse their thinking by creating the equations that generate mystery graphs. **<u>INTRODUCING THE GAME</u>** • Start by introducing the goal of the game: "In this game, you will use what you have learned about the effects of coefficients and constants on the graphs of quadratic equations to write equations that describe mystery graphs." • <u>Ask students to explain or rephrase the object of the game and the instructions</u>.* • Play one round of the game with you taking the role of Player 1; the class, Player 2. As you play, model the rules and the use of the recording sheet. • Create a mystery graph by entering a quadratic equation into Y1=. BE SURE NOT TO DISPLAY THE EQUATION TO THE CLASS. Display the "mystery graph" on the overhead to the class. Use the standard viewing window, when possible. Tell other player(s) if you have changed viewing window since $y= -x^2 – 10$ could look like $y= -x^2 – 5$ if you have changed the window to [-10, 10]. [-20, 20]. • Ask students to write down an equation for the graph. Students should share their equations with the members of their group. Each group should give a single equation for the mystery graph. Write the equations from all the groups on the board. • Select an *incorrect* equation from the list. Enter it into Y_2= on your calculator (again — do not display the Y= screen to the class) and display the graph of the proposed equation along with the graph of the mystery equation. • Demonstrate how to record this guess on the recording sheet. • Students should decide how to modify the equation if needed, and explain the rationale for their changes. • Repeat the process above until the class thinks they have found the correct equation. • Graph their equation on your calculator. Be sure to explicitly ask why there is only one graph on the screen, even though two equations were entered. • Display the Y= screen so that students can see that their equation matched the equation that created the mystery graph. • End by explaining how to award points for that round, i.e., 5 points for the correct equation, -1 point for each incorrect prediction. • Make clear the roles of Player 1 and Player 2 <u>Player 1</u>: Creates the mystery graph, enters Player 2's guesses into his or her calculator, then displays the graph of the guessed equation along with the mystery graph. <u>Player 2</u>: Tries to determine the equation of the mystery graph, recording his or her attempts on the recording sheet.

Phase	EXPLORE PHASE: Playing the game STRUCTURE
E X P L O R E E X P L O R E	**PARTNER WORK: PLAY THE GAME** • Each player should have the opportunity to write the equations for three mystery graphs. • As students play the game, circulate around the room. o Be sure students understand their role in playing the game. o Prompt students to use what they learned about coefficient and constants when creating equations. o Encourage students to modify their guessed equation based on the comparison of their graph to the mystery graph instead of starting over again each time. o Remind students to use the coefficients and constants from the list provided. o Encourage struggling players (e.g., those who appear to be guessing randomly) to graph his/her guess in his/her own calculator as a way to get started. o Be persistent in asking students to use appropriate mathematical language. o <u>Be sure to model appropriate mathematical language</u>.* <u>What do I do if students finish early?</u> Pairs can continue to play the game until all students have the chance to guess three mystery graphs or until time runs out.
	MONITORING STUDENTS' RESPONSES As you circulate, attend to students' mathematical thinking (as evidenced on their recording sheet and in their conversations). Use this as an opportunity to assess the extent to which students are able to use what they learned about coefficients and constants to write equations for specific graphs.

Ask the questions using everyday language (i.e., above or below) and then link the mathematical terms (i.e., vertical translation) to students' conceptualizations.

Phase	LESSON TASK QUESTIONS 1 - 4 How did you revise your predicted equation if the graph: 1. Opened in the wrong direction? 2. <u>Did not have the correct position above or below the x-axis</u>* (i.e., vertical translation)? 3. <u>Did not have the correct position to the right or left of the y-axis*</u> (i.e., horizontal translation)? 4. Was narrower or wider than the mystery graph?			
E X P L O R E	**Possible Solutions**	**Possible Questions**	**Misconceptions/Errors**	**Questions to Address Misconceptions/Errors**
	See discussion of Investigation 1 and 2.			

Share, Discuss, and Analyze - Investigation 3: *Name That Graph*

Orchestrating the mathematical discussion: a possible Sequence for sharing student work, Rationale and Mathematical Ideas to achieve the goals of the lesson, and possible Student Responses that demonstrate understanding.

The purpose of this sharing/discussion is to make explicit the strategies that students used in writing equations for mystery graphs.

Phase	Sequencing of Student Work	Rationale and Mathematical Ideas	Possible Questions and Student Responses
SHARE DISCUSS ANALYZE	This is a teacher-led discussion that reinforces the learning from Investigations 1 and 2. Students should give examples from their experience playing the game about the effects of coefficients and constants on graphs of quadratic equations. Don't belabor the debriefing of the game. All the ideas in the lesson will be addressed in the closing discussion (Wrap-Up).	The mathematical ideas are described in Investigations 1 and 2.	Go through the four questions, asking for examples of the modifications they made when their graph was: • Too narrow • Too wide • Opened in the wrong direction • Shifted the wrong direction, etc.

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WRAP - UP

Pulling it All Together

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Share, Discuss, and Analyze – Wrap-Up

Orchestrating the mathematical discussion: a possible Sequence for sharing student work, Rationale and Mathematical Ideas to achieve the goals of the lesson, and possible Student Responses that demonstrate understanding.

Revisiting the Mathematical Goals of the Lesson:
- Recognize the characteristic shape of a quadratic function and describe its maximum or minimum points (vertex).
- Develop an understanding of the relationship between quadratic equations and their graphs.
- Develop an understanding of the effect of coefficients and constants in quadratic equations on the appearance and position of their graphs.
- Predict from quadratic equations whether quadratic functions have maximum or minimum values and their location on the coordinate plane.
- Write quadratic equations to describe specific graphs.
- Reason mathematically and use and make connections among a variety of mathematical representations.

Revisiting the Academic Language Goals of the Lesson:
- Develop academic vocabulary to be used in the descriptions.
- Describe algebraic patterns orally or in writing.
- Explain the process used in solving the task, orally or in writing.
- Use appropriate mathematical language in all explanations and discussions.*

Phase	Sequencing of Student Work	Rationale and Mathematical Ideas	Possible Questions and Student Responses
S H A R E D I S C U S S A N A L Y Z E	This is a teacher-led discussion of the whole task. Students should be asked to look over their work on the task and to summarize their thinking. Before beginning the whole-group discussion, give students an opportunity to think about questions 4, 5, 6 and 7 and discuss them with their group. Questions 1, 2 and 3 are simply recaps of what has already been discussed in the investigations and game.	Students should explain their thinking to each question using examples from the investigations and game. There should also be an encore of the discussion of the effect of the values of *a*, *c*, *h* and *k* on the graphs of quadratic equations.	This phase is the debriefing of the entire task. The "Reflections" are the framing questions for the class discussion. 1. How does the value of *a* affect the appearance and location of the graph? (see previous discussion) 2. How does the value of the constant term *c* or *k* affect the appearance and location of the graph? (see previous discussion) 3. How does having an $(x + h)^2$ term (Investigation 2) affect the appearance and location of the graph? How do the graphs of these functions compare to those of Investigation 1? (see previous discussion)

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Phase	Sequencing of Student Work	Rationale and Mathematical Ideas	Possible Questions and Student Responses
S H A R E D I S C U S S A N A L Y Z E		Questions 4 and 5 extend what students learned about graphs of functions that open up or down to maximum and minimum values. Question 6 explicitly addresses the y-intercept. Note that the y-intercept is NOT readily apparent from equations written in the form $a(x+h)^2 + k$. It is not necessary to go into more depth about this here, because it will be revisited in Unit 4. Question 7 explicitly addresses the possible misconception or over-generalization about *c*, (i.e., that *c* always represents the vertical translation of the graph). While this is true of equations in the general form if b = 0 (i.e., for $y = ax^2 + c$), it is not true in general. Again, this concept will be revisited in Unit 4. The purpose of this discussion is simply to ensure that students don't leave this lesson with a misconception or over-generalization.	4. Can a quadratic function have ***both*** a maximum and minimum point? Why or why not? *No. A quadratic function has only a maximum or a minimum. The vertex is the maximum or minimum value (the "turn-around" point). A parabola (the graph of a quadratic function) has only one vertex.* 5. How can you tell from the equation whether a quadratic function has a maximum or minimum point on the graph? *Check the coefficient of the x^2 term. If it is positive, the graph opens up, so it has a minimum value. If it is negative, the graph opens down, so it has a maximum value.* 6. Does the graph of every quadratic function have a y-intercept? How can you determine the y-intercept by looking at the equation? *Yes, it always has a y-intercept. The graph keeps going forever, so at some point, one "arm" of the graph will intersect the y-axis. You can find the y-intercept by using x=0 and solving for y.* 7. Consider the equation: $y = x^2 - 4x + 7$. Todd says, "The graph has a minimum point of (0, 7) because c = 7." Lynn says, "I don't think so; c is equal to 7, but there's an "x" term in the equation. I think that might make a difference." Who is correct? Use your graphing calculator to investigate. *Lynn is correct; when there is an* x *term (i.e., $b \neq 0$), the* c *doesn't give the vertical translation. However,* c *is always the y-intercept.*

CLOSURE
Have students reflect on the mathematics of the lesson; find links to math that they have explored before; think of tasks that might be related to the big ideas of the lesson. In particular, compare what they learned about quadratic functions to what they know about linear functions. They should also be able to relate what they learned from exploring quadratics on the graphing calculator to what they learned when exploring the relation between the graphs linear functions and their equations.

Ask students to respond to the following prompts:
- **What do I know about quadratic functions: Make a list of those ideas you understand well.**
- **What questions do I have after the work that we have done? What should I do to help me answer these questions?**
- **What else do I want to know about quadratic functions, e.g. how do we use quadratic functions in real-life.**

It is important for students to step back and reflect on the ideas that surfaced and to situate their learning within past experiences, and to think forward to ways that they might build on these ideas in future tasks. This helps them to focus on the interconnectedness of mathematical ideas.

POSTERS
Groups can create posters to summarize what they learned about the effect of *a*, *c*, *h*, and *k* on the various forms of the quadratic. The first group to complete the work can create a poster for form $y + a(x + h)^2 + k$, the next group for form $y = a(x + h)^2$, etc. The posters will provide a record of their conclusions and can be referred to when quadratics are revisited in Unit 4.

CLOSURE
- Pose the following question and allow time for students to discuss it in small groups:
  Think about what you have learned about quadratic functions and their graphs. How are quadratic functions similar to, and different from, linear functions and their graphs? Create a "similarity/difference" table to record your responses.*

	Linear Functions	**Quadratic Functions**
Similarities		
Differences		