Squaring Triangles

Concept Lesson Guide

LESSON OVERVIEW

Overview: This lesson provides an opportunity for students to create four proofs of the Pythagorean Theorem. It assumes that students already can state that the theorem as $a^2 + b^2 = c^2$ where a and b are the legs of a right triangle and c is the hypotenuse. In "Getting Started," students review what they know about the Pythagorean Theorem. *Students who can not recall this prerequisite knowledge need a lesson about the theorem itself before beginning this lesson.*

In "Investigation 1," students establish three proofs of the theorem based on the same diagrams: one is a visual proof; the other two are algebraic proofs. In "Investigation 2," students establish a fourth proof of the theorem, also algebraic, based on a different diagram. See the teacher notes beginning on page 12 for a detailed discussion of the various proofs. Students will be expected to make conjectures about the shapes and to prove their conjectures based upon the given information and their knowledge of the properties of shapes and angles. They will then work together to develop a verbal and/or written proof of the Pythagorean Theorem based upon their observations and their knowledge of the algebraic relationship that they are trying to prove.

The whole class discussion should occur after students have completed both investigations.

NCTM Standards Addressed:

- Create and critique inductive and deductive arguments concerning geometric ideas and relationships, such as congruence, similarity, and the Pythagorean relationship.
- Build new mathematical knowledge through problem solving.
- Make and investigate mathematical conjectures.
- Develop and evaluate mathematical arguments and proofs.

Mathematical Goals of the Lesson:

- Proof can be algebraic, verbal, or visual.
- Create a visual proof of the Pythagorean Theorem using an area argument.
- Create algebraic proofs of the Pythagorean Theorem using an area argument.
- Use theorems involving supplementary, complementary and vertical angles to establish that given figures are squares.
- Develop algebraic-geometric reasoning skills.
- Proof requires the construction of a logical argument that builds on the given conditions and provides a valid justification for each statement in the argument.
- Students will reason mathematically and use and make connections among a variety of mathematical representations.
- Students will represent selected irrational numbers as the lengths of sides of right triangles and/or as locations on a number line. (Homework)

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Academic Language Goals of the Lesson:

- Develop academic vocabulary to be used in the proofs.
- Describe the relationships in a situation algebraically, orally or in writing.
- Explain the process used in solving the task, orally or in writing.

Assumption of Prior Knowledge:

- Knowledge of the Pythagorean theorem: $a^2 + b^2 = c^2$ where a and b are the legs of a right triangle and c is the hypotenuse.
- Definition of vertical angles and that vertical angles are congruent.
- Definition of supplementary and complementary angles and that their measures total 180° and 90° respectively.
- Area formulas for squares and triangles.
- Basic algebra skills, i.e., simplifying equations, multiplying binomials.
- Necessary and sufficient conditions to determine that a shape is a square.
- Definition of congruent figures and that corresponding parts of congruent figures are congruent.
- The sum of the angles of a triangle is 180°.
- Areas of congruent figures are equal, and that when equal areas are subtracted or added to congruent figures, the remaining or resulting areas are also equal.

Academic Language:

- Pythagorean Theorem
- Right triangle
- Leg
- Hypotenuse
- Supplementary angles
- Complementary angles
- Straight angles
- Corresponding parts of polygons
- Area
- Congruent
- Proof
- Irrational numbers (homework)
- Terminating and repeating decimals (homework)

Materials:

- Task
- Chart paper and markers, or
- Transparencies and pens

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Key:
Suggested teacher questions are shown in bold print.
Questions and strategies that support ELLs are underlined and identified by an asterisk.*
Possible student responses are shown in italics.

Phase	SET-UP PHASE: Setting Up the Mathematical Task
S E T U P	**INTRODUCING THE TASK** • Ask a student to read the introduction and Getting Started out loud as others follow along. Give students about s to do the Getting Started in their groups. • The teacher should create a master class list on transparency, chalkboard, or chart paper with facts the students recall about the Pythagorean Theorem. Consider asking each group to give one fact, then going to the next group, etc., until all the facts are listed. Be sure that the facts listed include: the theorem applies to any right triangle (and only right triangles); the theorem states that $a^2 + b^2 = c^2$ where *a* and *b* are the LENGTHS of the legs and *c* is the LENGTH of the hypotenuse; the hypotenuse is opposite the right angle and the legs are adjacent to the right triangle. Have a student draw and label a right triangle using *a, b, c*, stressing that these are the LENGTHS of the sides, not labels. • Ask a student to read Investigation 1 out loud as others follow along. • Clarify any confusions students may have, but do not suggest a method for solving the problems. Ask for any words or terms that the students do not understand. Clarify the meanings so that the task will be accessible to all students.* • Ask several students to explain what they are being asked to do in their own words.* • Specifically ask, "What is one thing you notice about Figure 1? What would you have to do prove that? What conditions would have to be true?" The purpose of these questions is to clarify what students are to do. Do not take the time to establish with the entire class that Figure 1 is a square. That is for the students to figure out. • Ask a student to read Investigation 2 out loud as others follow along. Again, ask another student to explain what they are being asked to do in their own words in this investigation. Because Investigation 2 is similar to Investigation 1, there is no need for further investigation of Investigation 2 at this time.

To assist ELLs' participation in the class discussion:*

- Allow time for students to first talk in small groups (pairs) and then have the groups report to the whole class.*
- Reinforce appropriate language as students communicate their ideas (e.g., revoice a student's contribution in mathematically and grammatically correct language). Ask students if you have captured what they said.*
- Develop a word wall and insert new terminology as it is introduced. Continually refer to it until the words become a part of student dialogue.*
- Expect students to incorporate appropriate mathematical vocabulary in their discussions once the terminology has been introduced.*

Phase	EXPLORE PHASE: Supporting Students' Exploration of the Task STRUCTURE
E X P L O R E	**<u>PRIVATE THINK TIME</u>** • Students work individually for about 10 minutes so that they can make sense of the problem for themselves. • Circulate around the classroom and clarify confusions. Be careful to NOT give away too much information or suggest a way to solve the problem.
E X P L O R E	**<u>SMALL-GROUP WORK</u>** • After about 10 minutes, ask students to work in their groups to discuss and compare what their individual answers to Questions 1 - 4. Groups that have difficulties with Questions 1 and 2 may want to work in pairs, with half the group working on Question 1 and the other Question 2, then compare results before attempting to go on to Questions 3 – 4. • As students are working, circulate around the room. o Be persistent in asking questions related to the mathematical ideas, problem-solving strategies, and connections between representations. o Be persistent in asking students to explain their thinking and reasoning. o Be persistent in asking students to explain, in their own words, what other students have said. o Be persistent in asking students to use appropriate mathematical language (e.g., complementary, supplementary, vertical, right and congruent) and be explicit about how they know that Figures 1 and 2, and shapes L, M and N are squares o Be persistent in asking students to use correct mathematical expressions for the formulas they generate to describe the relationships between the areas of the figures. • When students complete Investigation 1, they should continue on to Investigation 2. <u>What do I do if students have difficulty getting started?</u> • Have them consider Figure 2 first, because it is visually simpler. Ask, "What does the given information tell us about Figure 2? About its angles?" "What do we know about the lengths of its sides? How are the lengths of the sides of the triangles related to the lengths of the sides of the large quadrilateral in Figure 2? Can you label the diagram to show this?" <u>What do I do if students finish early?</u> • Look at students' justifications about Figures 1, 2, and 3 and shapes L, M, and N to be sure the reasons given are mathematically correct. • Examine students' proofs of the theorems to be sure the reasons given are mathematically correct. In particular, be sure that they are not assuming that the theorem is true while they are proving it (i.e., asserting that because the triangles are right triangles, they know that $a^2 + b^2 = c^2$). • Have them write up one or more of their proofs on chart paper or transparencies. • If they finish Investigation 2, they can continue on to the homework. • Ask a "quiet" group member to explain the work of the group. If s/he can't explain, challenge the group to make sure that each group member understands, leave, and return later to ask again.

Phase	EXPLORE PHASE: Supporting Students' Exploration of the Task STRUCTURE (continued)
E X P L O R E	**<u>MONITORING STUDENTS' RESPONSES</u>** • As you circulate, attend to students' mathematical thinking and to the strategies and representations used, in order to identify those responses that will be shared during the Share, Discuss, and Analyze phase. During this phase, groups will describe how they established that Figures 1* and 2* are congruent squares, that they have equal areas, and that shapes L, M and N are squares, and how they used this information to generate a visual proof and an algebraic proof of the Pythagorean Theorem. Similarly, in Investigation 2, they will prove that Figures 3 and P are squares, and find two different representations for the area of Figure 3.* • For Investigation 1, you will need to: o Identify students who can correctly explain and demonstrate that Figures 1* and 2* are congruent squares, that L, M and N are squares with areas a^2, b^2, and c^3, respectively,. Look for groups with different strategies for doing this. o Identify students who can explain the visual proof using an area argument. o Identify students who can explain the algebraic proof using an area argument based on Figures 1 and 2. o Identify students who can explain the algebraic proof using an area argument based on Figure 2. o Identify a group member of several different groups that can explain the answers to each question. • For Investigation 2, you will need to: o Identify students who can correctly explain and demonstrate that Figures 3* and P are squares, that P has side length b - a, and area $(b - a)^2$. Look for groups with different strategies for doing this. o Identify students who can algebraically represent the area of Figure 3* in two different ways, then create an equation and complete the proof. o Identify a group member of several different groups that can explain the answers to each question.

* Note: In most cases in our discussion, "Figure 1, Figure 2, and Figure 3" refer to the large quadrilaterals in each figure. However, when talking to students be clear when you are referring to the outside shape and the individual shapes contained within each of these quadrilaterals.

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Investigation 1- Proving the Pythagorean Theorem: There's More than One Way to Square a Triangle!

<table>
<tr><th>Phase</th><th colspan="4">TASK QUESTIONS 1 - 3</th></tr>
<tr><td></td><td colspan="4">1. Using the information that you have been given above, what can you conclude about the shapes and areas in Figures 1 and 2? Try to provide a justification for each of your conclusions.
2. Key Question: Use the information that you have established to construct a proof of the Pythagorean Theorem. Hint: both visual proofs and algebraic proofs are possible.
3. The Hindu mathematician Bhaskara (1114-1185) developed several proofs of the Pythagorean Theorem. He developed one from Figure 2 alone. Use what you established about the shapes in Figure 2 to construct a proof of the theorem.</td></tr>
<tr><td rowspan="2">E X P L O R E

E X P L O R E

E X P L O R E</td><th>Possible Solutions</th><th>Possible Questions</th><th>Misconceptions/ Errors</th><th>Questions to Address Misconceptions/Errors</th></tr>
<tr><td>Look for indictors of students' understanding:
• States that Figures 1*, 2*, L, M, and N are squares.
• Correctly proves that Figures 1*, 2*, L, M, and N are squares.
• Correctly determines the side length and the area of each square.
• Recognizes that Figures 1 and 2 are congruent and have equal areas.
• Recognizes that both Figure 1 and Figure 2 contain 4 congruent right triangles.
• Correctly represents the area of each square and right triangle algebraically.
• Creates two different correct algebraic expressions for the area of Figure 2.
• Writes a correct equation relating the areas of Figures 1 and 2.
• Writes a correct equation relating the two different expressions for the area of Figure 2.
• They can provide a valid justification for each of their conjectures.
See the Teacher Notes on Page 12 for a discussion of possible solution paths.</td><td>Ask questions such as:
• What have you come up with?
• What shape does _______ appear to be?
• What reasons can you give to support your conjecture?
• What are the properties of this shape?
• What do you know about the measure of the angles?
• How can you use what you know to find the measures of the other angles?
• How can you find the length of the sides?
• How are the lengths of the sides of the squares related to those of the right triangle?
• If you know the side lengths, what else can you determine?
• How are Figures 1 and 2 alike? How are they different?
• Look at the theorem you're trying to prove. How are the variables related? How could you represent that geometrically?
• Explain in your own words what _ _______(another student) said.*</td><td>• Thinking that you can establish a shape is a square because it looks like one.
• Assuming the relationships in the theorem are true instead of proving them. (i.e., the area of L and M together equal N because a, b, and c are the side lengths of a right triangle.
• Not recognizing the connection between the algebraic representation (a^2) and geometric representation (the area of a square with side length a).</td><td>• How can you justify mathematically that figure __________ is a square?
• What evidence do you have about its sides and angles?
• What is the length of each side?
• How do you know that is true? What the givens? What are you trying to prove?
• How are Figures 1 and 2 related? How can you describe that algebraically?
• How can you mathematically justify each step in your proof?
• Look at the theorem you're trying to prove. How are the variables related? How could you represent that geometrically?</td></tr>
</table>

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Investigation 2- A Chinese Proof

<table>
<tr><th>Phase</th><th colspan="4">TASK QUESTIONS 1 – 2
1. What can you conclude about Figure 3 and the shapes in it?
2. How can this figure be used to prove the Pythagorean Theorem?</th></tr>
<tr><td></td><td>Possible Solutions</td><td>Possible Questions</td><td>Misconceptions/ Errors</td><td>Questions to Address Misconceptions/Errors</td></tr>
<tr><td>EXPLORE
EXPLORE
EXPLORE</td><td>Look for indictors of students' understanding:
• States that Figures 3* and P are squares.
• Correctly proves that Figures 1*, 2, L, M, and N are squares.
• Correctly determines the side length and the area of Figure 3.
• Recognizes that Figures 1 and Figure 2 are congruent and have equal areas.
• Recognizes that both Figure 1 and Figure 2 contain 4 congruent right triangles.
• Correctly represents the area of each square and the right triangle algebraically.
• Creates two different correct algebraic expressions for the area of Figure 3.
• Writes a correct equation relating the areas of Figures 1 and 2.
• Writes a correct equation relating the two different expressions for the area of Figure 2.
• They can provide a valid justification for each of their conjectures.

See the Teacher Notes on Page 12 for a discussion of possible solution paths.</td><td>Ask questions such as:
• What have you come up with?
• What shape does _______ appear to be?
• What reasons can you give to support your conjecture?
• What are the properties of this shape?
• What do you know about the measure of the angles?
• How can you use what you know to find the measures of the other angles?
• How can you find the length of the sides?
• How are the lengths of the sides of the squares related to those of the right triangle?
• If you know the side lengths, what else can you determine?
• What is one way to represent the area of Figure 3?
• How can you use the length of the sides to represent area?
• Is there another way to represent the area of Figure 3?
• Explain in your own words what ______(another student) said.*</td><td>• Thinking that you can establish a shape is a square because it looks like one.

• Not recognizing that P has side length b - a.

• Not recognizing the connection between the algebraic representation (a^2) and geometric representation (the area of a square with side length a).</td><td>• How can you justify mathematically that Figure __________ is a square?

• What evidence do you have about its sides and angles?

• What line segments in the figure do you know the lengths of?

• How can you use the lengths that you know to represent the ones you don't know?

• How is the length of the side of P related to the legs of the right triangle?</td></tr>
</table>

Share, Discuss, and Analyze

Orchestrating the mathematical discussion: a possible Sequence for sharing student work, Rationale and Mathematical Ideas to achieve the goals of the lesson, and possible Student Responses that demonstrate understanding.

Mathematical Goals of the Lesson:

- Proof can be algebraic, verbal, or visual.
- Create a visual proof of the Pythagorean Theorem using an area argument.
- Create algebraic proofs of the Pythagorean Theorem using an area argument.
- Use theorems involving supplementary, complementary and vertical angles to establish that given figures are squares.
- Develop algebraic-geometric reasoning skills.
- Proof requires the construction of a logical argument that builds on the given conditions and provides a valid justification for each statement in the argument.
- Students will reason mathematically and use and make connections among a variety of mathematical representations.
- Students will represent selected irrational numbers as the lengths of sides of right triangles and/or as locations on a number line. (Homework)

Phase	Sequencing of Student Work	Rationale and Mathematical Ideas	Possible Questions and Student Responses
SHARE DISCUSS ANALYZE	Each group (or select groups) describes one of their proofs, including their justifications of the initial conditions, i.e., that that Figures 1*, 2*, 3*, L, M, N and are squares. If possible, for each proof, select at least two groups whose proofs differ in how they established the initial conditions. Proofs should be presented in the order in the investigation, visual proof, followed by the algebraic proofs in Investigations 1 and 2 respectively.	This discussion will establish four proofs of the Pythagorean theorem. It is also an opportunity to explicitly discuss that a proof can be visual; it doesn't have to be algebraic or verbal. The relationships between the diagrams and the algebra should be discussed and demonstrated reinforcing geometric-algebraic reasoning.	**What did you conclude about Figure 1 and shapes L and M?** • *They are all squares.* **How can you prove they are squares?** (Answers will vary) • *All the angles in the shapes formed by two triangles are right angles (due to complementary angles). L and M are squares because they have four right angles (using supplementary angles) and four congruent sides.* • *L and M are rhombi because they have four congruent sides. They also have two right angles (supplementary angles). Because opposite angles of rhombi are congruent, L and M have four right angles, so are squares.* **What did you conclude about Figure 2 and shape N?** • *Figure 2 is a square because it has four right angles and all its sides are equal to a + b.* • *N is a square. All of its sides have length c. All its angles are 90° because each corner angle of N, plus the angles of the two triangles is 180. The two angles of the triangles total 90°, so each corner of N also has to be 90°.* **How are Figures 1 and 2 related?** • *The figures are congruent because both are squares with sides a + b. They also both contain 4 congruent right triangles. When you take the 4 right triangles away from each figure, you're left with L and M of Figure 1 and N from Figure 2. They must be equal because you took the same amount from both large squares. So the area of L plus the area of M is equal to the area of N.*

Phase	Sequencing of Student Work	Rationale and Mathematical Ideas	Possible Questions and Student Responses
S H A R E D I S C U S S A N D A N A L Y Z E			**How is this related to the Pythagorean Theorem?** (Ask this for each proof.) • *a and b are the lengths of the legs of the right triangle, c is the length of the hypotenuse. Therefore , or* $a^2 + b^2 = c^2$*. So, we showed that the sum of the squares of the legs = square of hypotenuse.* **How can you show that algebraically?** *The area of Figure 1 is of* $a^2 + b^2 + 4*(1/2)ab$ *The area of Figure 2 is* $c^2 + 4*(1/2)ab$ *Setting them equal to each other and subtracting 4*(1/2)ab from both sides gives* $a^2 + b^2 = c^2$ (You could also ask "**Where did the** a^2 **come from",** etc.) **What is the relationship between the visual proof and the algebraic proof?** • *They are the same. In both you subtract the area of the 4 congruent right triangles from the two congruent squares, showing that what is left is equal.* Ask similar questions about Proofs #3 and #4. In Proof 4, be sure to ask how the students know that the length of the side of square P is b-a. (Methods to establish that Figures 3 and P are squares are similar to those used for the figures in Investigation 1). **Look at Proofs 3 and 4. How are they similar? How are they different?** • *They are similar because in both, you write two different expressions for the area of a square, set them equal to each other, then simplify the equations.* • *They differ in the position of the right triangles. So, in Figure 2, N has side length c, which is easy to see. In Figure 3, the side length of P is not a, b or c, so is not as easy to figure out.*

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Share, Discuss, and Analyze the Task - WRAP-UP: Pulling it All Together

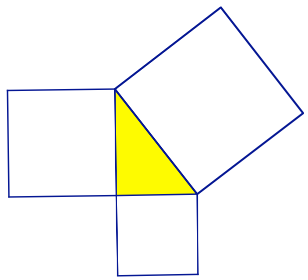
CLOSURE

Have students reflect on the mathematics of the lesson and the different proofs they established.

"Which of the proofs did you find easiest to follow? The most difficult to understand? Why?"

"Look at all four proofs. Are there any general strategies that were used in all, or most of the proofs that might be useful in the future?"

"How does the diagram on the task sheet relate to the Pythagorean Theorem?"



It is important for students to step back and reflect on the ideas that surfaced and to situate their learning within past experiences, and to think forward to ways that they might build on these ideas in future tasks. This helps them to focus on the interconnectedness of mathematical ideas.

HOMEWORK

Wheel of Theodorus assignment, at the end of the task.

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Extension: Other Proofs of the Pythagorean Theorem
Watch Theorem of Pythagoras video (T. Apostol series, Cal Tech)
This video shows a number of different proofs. Consider focusing on the sheer proof. Discuss sheers; explain why this works, etc.

Further Extensions
Special triangles
Finding all possible squares on geoboard and their areas.
Establishing the distance formula.

Teacher Notes:

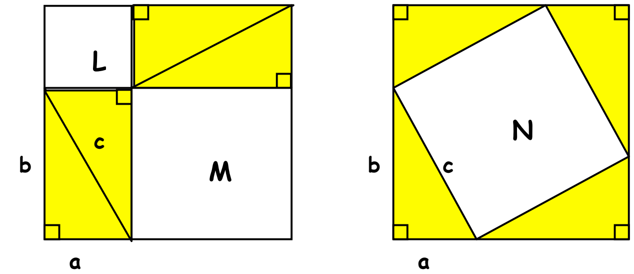

Figure 1 Figure 2

Here are the three proofs that students can establish in this lesson:

Givens: All the right triangles are congruent, with legs a and b, and hypotenuse c; the outside borders of the large squares are straight lines.

First students must establish that Figures 1 and 2 are congruent squares. They must also establish that Figures L, M and N are squares with side lengths a, b, and c respectively. (There are a variety of ways to do that, using the given information as well as vertical, supplementary, and complimentary angles, and properties of squares.)

Proof 1: Visual proof: Figures 1 and 2 have the same area, because they are congruent. Both figures contain four congruent right triangles. If these triangles are removed from each figure, the remaining areas of Figures 1 and 2 (the white regions) are equal. That is, the area of M + area of L = area of N, or $a^2 + b^2 = c^2$.

Proof 2: This is an algebraic representation of the visual proof based on Figures 1 and 2:

$a^2 + b^2 + 4*(1/2)ab = c^2 + 4*(1/2)ab$

Subtracting 4*(1/2)ab from both sides gives

$a^2 + b^2 = c^2$

Proof 3: This proof is based on creating two different representations for the area of Figure 2: $(a + b)^2$ (the square of the side length a + b) and $c^2 + 4*(1/2)ab$ (the area of N + the areas of the four triangles). Setting these equal and doing the algebra produces:

$(a + b)^2 = c^2 + 4*(1/2)ab$

$a^2 + 2ab + b^2 = c^2 + 2ab.$

Subtracting 2ab from both sides gives: $a^2 + b^2 = c^2$

Proof 4: Givens: All the right triangles are congruent, with legs a and b, and hypotenuse c.

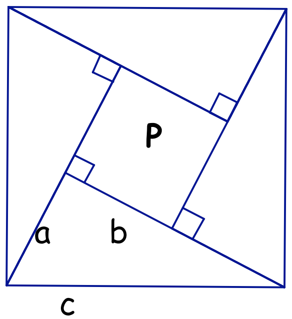

Figure 3

This proof is based on creating two different representations for the area of the outside square in Figure 3: c^2 (the outside square with side length "c") and the area of square P + the areas of the four triangles. (alternatively they can create two different representations for the area of square P).

First students must establish that P and the outer quadrilateral are squares. There are a variety of ways to do that, using vertical, supplementary, and complimentary angles, and properties of squares.

Then, students must establish that square P has side length b – a, so its area is $(b – a)^2$.

Option 1: Two representations for the area of the large square can be created: c^2 and $(b – a)^2 + 4*(1/2)ab$.

Setting these equal and performing algebraic manipulations produces:

$c^2 = (b – a)^2 + 4*(1/2)ab$

$c^2 = b^2 - 2ab + a^2 + 2ab = b^2 + a^2$

So, $c^2 = b^2 + a^2$

Option 2: Alternatively two representations for the area of P can be created: $(a - b)^2$ and $c^2 – 4(1/2)ab$.

Expanding the first expression, setting these equal and performing algebraic manipulations produces:

$a^2 - 2ab + b^2 = c^2 – 2ab$ thus $c^2 = b^2 + a^2$

Implementation suggestion:

1) Introduce the task, discuss the Getting Started, and introduce Investigations 1 and 2. Let students do both investigations in their groups, monitoring group interaction. Note that the proof in Investigation 2 is similar to the third one in Investigation 1 in that it involves creating two different algebraic representations for the area of the given figure, setting them equal to each other, then simplifying the resulting equation. Have each group record at least one of their proofs, including how they proved that Figures 1, 2 and shapes L, M, and N are squares, either on transparency, chalkboard or chart paper to explain during the class discussion.
2) After students have completed both investigations, bring the groups together for a whole class discussion of the questions and presentation of their proofs.

Alternatively, if the whole class appears to be struggling, the teacher may want to have a class discussion of the first proof to help everyone understand the expectations and possible ways to proceed.